Cosmological Models in Gravitational Scalar-Tensor Theories

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Abstract. In this work, a brief review of a new form of scalar-tensor theories of gravity, known as gravitational scalar-tensor theories in which the action is composed of the Ricci scalar and its first and second derivatives is made. Some of the cosmological applications in these new theories are discussed considering different models corresponding to the first non-trivial extensions of general relativity possessing 2 + 2 degrees of freedom. We show that the resulting cosmological behavior is in agreement with observations.

1. Introduction

Despite the success of the standard cosmological model based on the theory of General Relativity (GR), the recent developments in observational cosmology and astronomy led to the statement that this model is inadequate to explain many phenomena in the Universe [1]. The Universe has experienced two phases of cosmic acceleration. The first one is the inflation phase which is an early-time accelerated phase and it is believed to have happened a fraction of second after the Big Bang and prior to the radiation-dominated epoch. The second phase is the accelerated expansion in the present universe which is considered to have started since the Universe entered its dark-energy-dominated epoch [2]. The two phases of accelerated expansion are very challenging however, there are recent attempts and alternatives to the standard Big Bang model proposed to explain these phenomena. Some of those attempts are within the framework of GR and some focused on the possibility of modifying GR, by modifying the gravitational sector of the theory. Extending the work done in [3], this particular piece of literature is aimed at studying a new form of scalar-tensor theories of gravity, known as gravitational scalar-tensor theories. There has been some recent work on the scalar-tensor theories of gravity on how to construct a gravitational modification without presenting ghosts behavior or any instabilities which normally arise due to the extra degrees of freedom introduced in such modification. In 1974, Horndeski [4] was able to construct the single-scalar field theory with second-order equations of motion with respect to the scalar field and the metric which involves $2+1$ propagating degrees of freedom, and thus without ghosts. This theory is further extended as in [5–10] with the same $2+1$ propagating degrees of freedom. In [11], the authors managed to construct a gravitational modification namely gravitational scalar-tensor theories, which possess $2+2$ degrees of freedom that propagate without introducing ghost nor Ostrogradski instabilities [12] under a specific choice of the Lagrangian. They considered a theory of gravity with an action that is composed of the Ricci scalar and its first and second derivatives. [3, 11]

$$S = \frac{1}{2}\int d^4x\sqrt{-g}f\Big(R, (\nabla R)^2, \Box R\Big)\,, \tag{1}$$

where $\Box R = g^{ab}\nabla_a\nabla_b R$. The action of these theories can be transformed to an action of multi-scalar fields coupled to gravity, by using double Lagrangian multipliers. E.g., if we have $f(R)$ and $f(\phi)$, we introduce a new variable λ called a Lagrangian multiplier and it is defined $\mathcal{L}(R,\phi,\lambda) = f(\phi) - \lambda(\phi - R)$, so in case of $f(R, (\nabla R)^2, \Box R)$, we introduce a set of Lagrange

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multipliers $(\tilde{\lambda}, \tilde{\Lambda}_1, \tilde{\Lambda}_2)$ and the associated auxiliary fields (ϕ, X, B) in order to reduce the order of derivatives [3]:

$$f(R, (\nabla R)^2, \Box R) = f(\phi, X, B) - \tilde{\lambda}(\phi - R) - \tilde{\Lambda}_1(X - (\nabla R)^2) - \tilde{\Lambda}_2(B - \Box R) \ , \tag{2}$$

where

$$\tilde{\lambda} = \lambda + \nabla^\rho[\tilde{\Lambda}_1 \nabla_\rho(\phi + R)] - \Box\tilde{\Lambda}_2 \ , \quad \tilde{\Lambda}_1 = \Lambda_1 \ , \quad \tilde{\Lambda}_2 = \Lambda_2 \ . \tag{3}$$

By using Eq. (2) and Eq. (3) and by replacing all the derivatives of R with the derivatives of ϕ, one can rewrite the action in Eq. (1) as follows [11]

$$S = \int d^4x\sqrt{-g}\Big[f(\phi, X, B) - \lambda(\phi - R) - \Lambda_1(X - (\nabla\phi)^2) - \Lambda_2(B - \Box\phi)\Big] \ . \tag{4}$$

This action does not involve any derivative terms of those two variables Λ_1 and Λ_2 which implies that the variation of the action with respect to these two variables results in constraint equations [11]. Therefore, the action in Eq. (4) can be written as

$$S = \int d^4x\sqrt{-g}\Big[f(\phi, (\nabla\phi)^2, \Box\phi) - \lambda(\phi - R)\Big] \ . \tag{5}$$

This action has derivative terms of ϕ, which implies a dynamical equation of ϕ. Since the higher derivatives terms except for λR come from $\Box\phi$, therefore λ is also treated as a dynamical field. In order to reduce the order of derivatives, in [11], another Lagrangian multiplier and the associated auxiliary field have been introduced for $\Box\phi$ as

$$f(\Box\phi) = f(B) - \Lambda(B - \Box\phi) \ , \tag{6}$$

therefore, the action in Eq. (5) is now written as

$$S = \int d^4x\sqrt{-g}\Big[f(\phi, (\nabla\phi)^2, B) - \lambda(\phi - R) - \Lambda(B - \Box\phi)\Big] \ . \tag{7}$$

By varying this action with respect to B, given that this action is independent of the derivative of B [11], therefore we have

$$\delta S = \int d^4x\sqrt{-g}[f_B - \Lambda] \ . \tag{8}$$

Since the action is independent of the derivative of B, δS yields a constraint equation, $f_B = \Lambda$, where f_B denotes partial derivative of f with respect to B. By replacing this constraint back into the action in Eq. (7), and by varying the action with respect to B, the resulting action is

$$\delta S = \int d^4x\sqrt{-g}[-f_{BB}(B - \Box\phi)] \ . \tag{9}$$

1- If $f_{BB} = 0$, B does not enter linearly in the Lagrangian and under a conformal transformation where $g_{ab} = \frac{1}{2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\hat{g}_{ab}$, the action of the new gravitational scalar-tensor is given as [3]

$$S = \int d^4x\sqrt{-\hat{g}}\Big[\frac{1}{2}\hat{R} - \frac{1}{2}\hat{g}^{ab}\Big(\partial_a\mathcal{X}\partial_b\mathcal{X} + e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\partial_a\varphi\partial_b\phi\Big) \\ -\frac{1}{4}\Big(e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi + e^{-2\sqrt{\frac{2}{3}}\mathcal{X}}\Big(\varphi B(\phi, (\hat{\nabla}\phi)^2, \varphi) - f\Big)\Big)\Big] \ . \tag{10}$$

2- If $f_{BB} = 0$, B enters linearly in the Lagrangian, and the function f can be rewritten as

$$f\Big(R, (\nabla R)^2, \Box R\Big) = \mathcal{K}\Big(R, (\nabla R)^2\Big) + \mathcal{G}\Big(R, (\nabla R)^2\Big)\Box R \ . \tag{11}$$

The action of the new gravitational scalar-tensor is given by [3, 11]

$$S=\int d^4x\sqrt{-\hat{g}}\Big[\frac{1}{2}\hat{R}-\frac{1}{2}\hat{g}^{ab}\partial_a\mathcal{X}\partial_b\mathcal{X}-\frac{1}{\sqrt{6}}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\hat{g}^{ab}\mathcal{G}\partial_a\mathcal{X}\partial_b\phi$$
$$+\frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\mathcal{X}}\mathcal{K}+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\mathcal{G}\hat{\Box}\phi-\frac{1}{4}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi)\Big]\ , \tag{12}$$

where

$$\mathcal{K}=\mathcal{K}(\phi,B)\ ,\quad \mathcal{G}=\mathcal{G}(\phi,B)\ ,\quad B=2e^{\sqrt{\frac{2}{3}}\mathcal{X}}g^{ab}\partial_a\phi\partial_b\phi\ . \tag{13}$$

As shown in [11], despite the higher derivative nature of the Lagrangian, this new theory does not introduce any ghost under an appropriate choice of the Lagrangian.

2. Field equations in the gravitational scalar-tensor theories

In [3], the authors investigated the cosmological behavior in gravitational scalar-tensor theories to study the late-time evolution of a universe governed by these theories. They introduced the action of the matter sector S_m, and considered a homogeneous and isotropic geometry in the Einstein framework such that the total action is $S = S + S_m$. We consider the following flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime metric where the two scalar fields are time-dependent only:

$$ds^2=-dt^2+a(t)^2\delta_{ij}dx^idx^j\ . \tag{14}$$

Therefore, the Friedmann equations and the evolution equations are given as [3],

$$3H^2-\rho_m-\frac{1}{2}\dot{\mathcal{X}}^2+\frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\mathcal{X}}\mathcal{K}+\frac{2}{3}\dot{\phi}^2\Big(\dot{\phi}(\sqrt{6}\dot{\mathcal{X}}-9H)-3\ddot{\phi}\Big)\mathcal{G}_B$$
$$-\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\Big(\dot{B}\dot{\phi}\mathcal{G}_B+\frac{1}{2}\phi+\dot{\phi}^2(\mathcal{G}_\phi-2\mathcal{K}_B)\Big)=0\ , \tag{15}$$
$$3H^2+2\dot{H}+p_m+\frac{1}{2}\dot{\mathcal{X}}^2+\frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\mathcal{X}}\mathcal{K}+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\Big(\dot{B}\dot{\phi}\mathcal{G}_B-\frac{1}{2}\phi+\dot{\phi}^2\mathcal{G}_\phi\Big)=0\ , \tag{16}$$
$$\varepsilon_{\mathcal{X}}=\ddot{\mathcal{X}}+3H\dot{\mathcal{X}}-\frac{1}{3}\dot{\phi}^2\Big(\dot{\phi}(3\sqrt{6}H-2\dot{\mathcal{X}})+\sqrt{6}\ddot{\phi}\Big)\mathcal{G}_B$$
$$+\frac{1}{2\sqrt{6}}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\Big(2\dot{B}\dot{\phi}\mathcal{G}_B-\phi+2\dot{\phi}^2(\mathcal{K}_B+\mathcal{G}_\phi)\Big)+\frac{1}{\sqrt{6}}e^{-2\sqrt{\frac{2}{3}}\mathcal{X}}\mathcal{K}=0\ , \tag{17}$$
$$\varepsilon_\phi=\frac{1}{3}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\Big(\dot{\phi}\big(-9H+\sqrt{6}\dot{\mathcal{X}}\big)-3\ddot{\phi}\Big)\mathcal{K}_B+\frac{1}{6}\dot{B}\Big\{3e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\dot{B}+4\dot{\phi}\Big(\dot{\phi}(9H-\sqrt{6}\dot{\mathcal{X}})+3\ddot{\phi}\Big)\Big\}\mathcal{G}_{BB}$$
$$+\frac{1}{3}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\Big(\dot{\phi}(9H-\sqrt{6}\dot{\mathcal{X}})+3\ddot{\phi}\Big)\mathcal{G}_\phi\Big\{e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\dot{B}\dot{\phi}+\frac{2}{3}\dot{\phi}^2\Big(\dot{\phi}(9H-\sqrt{6}\dot{\mathcal{X}})+3\ddot{\phi}\Big)\Big\}\mathcal{G}_{B\phi}$$
$$-e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\dot{\phi}^2\mathcal{K}_{B\phi}+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\dot{\phi}^2\mathcal{G}_{\phi\phi}-e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\dot{B}\dot{\phi}\mathcal{K}_{BB}+\Big\{\frac{4}{3}\dot{\phi}(9H-2\sqrt{6}\dot{\mathcal{X}})\ddot{\phi}$$
$$-\frac{1}{\sqrt{6}}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\dot{B}\dot{\mathcal{X}}+\dot{\phi}^2\Big(18H^2+6\dot{H}-3\sqrt{6}H\dot{\mathcal{X}}-\frac{2}{3}\dot{\mathcal{X}}^2-\sqrt{6}\ddot{\mathcal{X}}\Big)\Big\}\mathcal{G}_B$$
$$-\frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\mathcal{X}}\mathcal{K}_\phi+\frac{1}{4}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}=0\ . \tag{18}$$

Here $B(t)=-2e^{\sqrt{\frac{2}{3}}\mathcal{X}}\dot{\phi}^2$, and dots denote differentiation with respect to time and the subscripts refer to the partial differentiations with respect to the corresponding argument. The Friedmann equations presented in Eqs. (15) and (16) can be written as

$$H^2=\frac{1}{3}(\rho_{DE}+\rho_m)\ ,\quad 2\dot{H}+3H^2=-(p_{DE}+p_m)\ , \tag{19}$$

where ρ_{DE} and p_{DE} represent an effective dark energy sector with energy density and pressure respectively:

$$\rho_{DE} = \frac{1}{2}\dot{\mathcal{X}}^2 - \frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\mathcal{X}}\mathcal{K} - \frac{2}{3}\dot{\phi}^2\Big(\dot{\phi}(\sqrt{6}\dot{\mathcal{X}} - 9H) - 3\ddot{\phi}\Big)\mathcal{G}$$
$$+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\Big(\dot{B}\dot{\phi}\mathcal{G} + \frac{\phi}{2} + \dot{\phi}^2(\mathcal{G}_\phi - 2\mathcal{K}_B)\Big)\,, \tag{20}$$
$$p_{DE} = \frac{1}{2}\dot{\mathcal{X}}^2 + \frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\mathcal{X}}\mathcal{K} + \frac{1}{2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\Big(\dot{B}\dot{\phi}\mathcal{G}_B + \dot{\phi}^2\mathcal{G}_\phi - \frac{\phi}{2}\Big)\,, \tag{21}$$

with the dark-energy equation of state parameter $w_{DE} = \frac{p_{DE}}{\rho_{DE}}$.

3. Cosmological applications in gravitational scalar-tensor theories

In [3], the authors have considered different functional forms of $\mathcal{K}$ and $\mathcal{G}$ to study the late-time evolution of a universe governed by these new theories considering different models. In this subsection, we will investigated the cosmological application of the new gravitational scalar-tensor by considering two different models by choosing particular functional form of $\mathcal{K}$ and $\mathcal{G}$.

Model 1: $\mathcal{K}(\phi, B) = \frac{\phi^2}{2} - \frac{\zeta}{2}B$ *and* $\mathcal{G}(\phi, B) = 0$

Here ζ is the corresponding coupling constant. We apply a transformation to any time derivative function f and H into a redshift z derivative as follows

$$\dot{f} = -H(1+z)H\frac{df}{dz}\,, \tag{22}$$
$$\ddot{f} = H(1+z)^2\frac{dH}{dz}\frac{df}{dz} + H^2(1+z)\frac{df}{dz} + H^2(1+z)^2\frac{d^2f}{dz^2}\,. \tag{23}$$

The redshift is given in terms of the scale factor as $z = (-1+\frac{a_0}{a})$, with the normalized coefficient represented by a_0 and consider the matter sector to be dust i.e., $p_m \simeq 0$. In this model, the Friedmann and the evolution equations Eqs. (15) - (18) and the effective dark-energy energy density and pressure Eqs. (20) - (21), in the redshift space become:

$$\mathcal{X}'' + \Big(\frac{1}{H}\frac{dH}{dz} - \frac{2}{(1+z)}\Big)\mathcal{X}' + \frac{1}{2\sqrt{6}}\zeta e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi'^2 - \frac{\Big(1 - e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi\Big)}{2\sqrt{6}H^2(1+z)^2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi = 0\,, \tag{24}$$
$$\zeta\phi'' + \zeta\Big(\frac{1}{H}\frac{dH}{dz} - \frac{2}{(1+z)}\Big)\phi' - \frac{\sqrt{6}}{3}\zeta\phi'\mathcal{X}' + \frac{1}{2H^2(1+z)^2}\Big(1 - e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi\Big) = 0\,, \tag{25}$$
$$\frac{1}{6}\Big(\mathcal{X}'^2 + \frac{1}{2}\zeta e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi'^2\Big)(1+z)^2 - \frac{2(1+z)}{3H}\frac{dH}{dz} - \frac{\Big(1 - \frac{1}{2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi\Big)}{12H^2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi = 0\,, \tag{26}$$
$$\Omega_m = 1 - \frac{1}{6}\Big(\mathcal{X}'^2 + \frac{1}{2}\zeta e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi'^2\Big)(1+z)^2 - \frac{1}{12H^2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\Big(1 - \frac{1}{2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi\Big)\phi\,, \tag{27}$$
$$q = \frac{3\Omega_m}{2} + \frac{1}{2}\Big(\mathcal{X}'^2 + \frac{1}{2}\zeta e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi'^2\Big)(1+z)^2 - 1\,, \tag{28}$$
$$\omega_{DE} = \frac{1}{3\Omega_{DE}}\Big(\mathcal{X}'^2 + \frac{1}{2}\zeta e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi'^2\Big)(1+z)^2 - 1\,, \tag{29}$$
$$\Omega_{DE} = \frac{1}{6}\Big(\mathcal{X}'^2 + \frac{1}{2}\zeta e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi'^2\Big)(1+z)^2 + \frac{1}{12H^2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\Big(1 - \frac{1}{2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi\Big)\phi\,. \tag{30}$$

We solve the whole system of equations for this model numerically and we present the cosmological evolution for the matter, dark energy density parameters, the dark-energy equation of state parameters, as well as the evolution of the deceleration in Figs. 1 - 4 for the parameters choice $\zeta = 10$, $\Omega_{m_0} = \frac{\rho_{m_0}}{3H^2} \simeq 0.3$ and $\Omega_{DE_0} = \frac{\rho_{DE_0}}{3H^2} \simeq 0.7$.

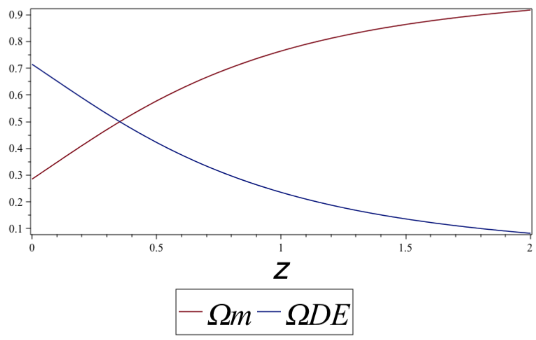

Figure 1. The evolution of Ω_m and Ω_{DE} versus the redshift z for model 1.

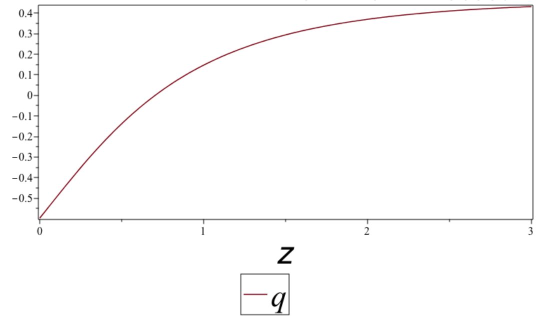

Figure 2. The evolution of the deceleration parameter q versus the redshift z for model 1.

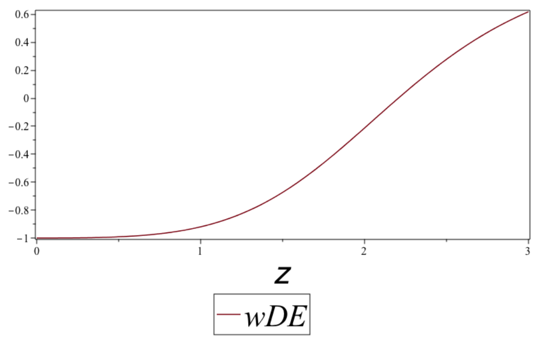

Figure 3. The evolution of the dark-energy equation of state parameter ω_{DE} versus the redshift z for model 1.

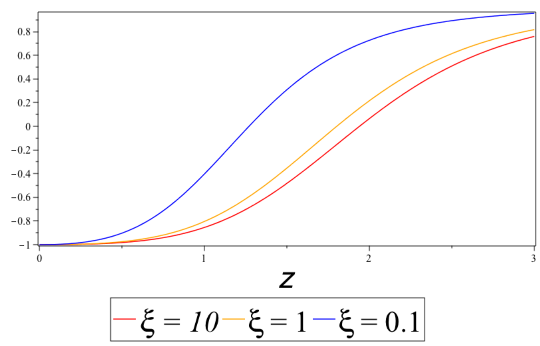

Figure 4. The evolution of ω_{DE} versus the redshift z for and different values of the model parameter ζ for model 1.

Model 2: $\mathcal{K}(\phi, B) = \zeta B$ *and* $\mathcal{G}(\phi, B) = \frac{\zeta\phi^2}{2}$
In this model, the Friedmann and the evolution equations Eqs. (15) - (18) and the effective dark-energy energy density and pressure Eqs. (20) - (21), in the redshift space become:

$$\mathcal{X}'' + \Big(\frac{1}{H}\frac{dH}{dz} - \frac{2}{(1+z)}\Big)\mathcal{X}' - \frac{1}{\sqrt{6}}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\zeta(1-\phi)\phi'^2 - \frac{1}{\sqrt{6}H^2(1+z)^2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi = 0\,, \quad (31)$$

$$\zeta\phi'' + \zeta\Big(\frac{1}{H}\frac{dH}{dz} - \frac{2}{(1+z)}\Big)\phi' - \frac{\sqrt{6}}{3}\zeta\phi'\mathcal{X}' + \frac{1}{4H^2(1+z)^2}\Big(1 - 2\zeta H^2(1+z)^2\phi'^2\Big) = 0 \quad (32)$$

$$-2H(1+z)\frac{dH}{dz} + 3H^2 + \frac{1}{2}\Big(\mathcal{X}'^2 - (1-\phi)\zeta e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi'2\Big)H^2(1+z)^2 - \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi = 0\,, \quad (33)$$

$$\Omega_m = 1 - \frac{1}{6}\Big(\mathcal{X}'^2 - (1-\phi)\zeta e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi'^2\Big)(1+z)^2 - \frac{1}{12H^2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi\,, \quad (34)$$

$$q = \frac{3\Omega_m}{2} + \frac{1}{2}\Big(\mathcal{X}'^2 - (1-\phi)\zeta e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi'^2\Big)(1+z)^2 - 1\,, \quad (35)$$

$$\omega_{DE} = \frac{1}{3\Omega_{DE}}\Big(\mathcal{X}'^2 + (1-\phi)\zeta e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi'^2\Big)(1+z)^2 - 1\,, \quad (36)$$

$$\Omega_{DE} = \frac{1}{6}\Big(\mathcal{X}'^2 - (1-\phi)\zeta e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi'^2\Big)(1+z)^2 + \frac{1}{12H^2}e^{-\sqrt{\frac{2}{3}}\mathcal{X}}\phi\,. \quad (37)$$

We solve the whole system of equations for this model numerically for the parameters choice $\zeta = -1$, we present the cosmological evolution for the matter, dark energy density parameters, the dark-energy equation of state parameters and the deceleration parameter in Figs. 5 - 7.

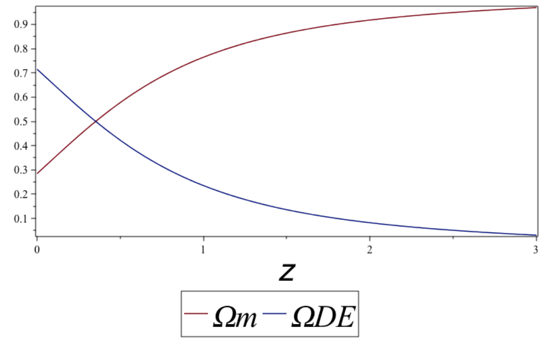

Figure 5. The evolution of Ω_m and Ω_{DE} versus the redshift z for model 2.

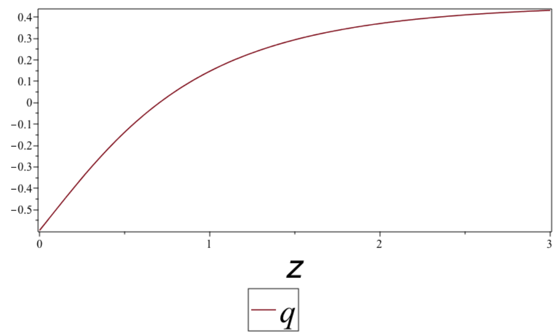

Figure 6. The evolution of the deceleration parameter q versus the redshift z for model 2.

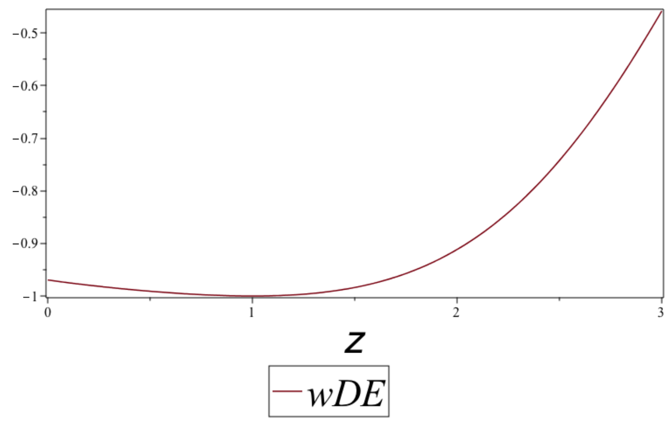

Figure 7. The evolution of the dark-energy equation of state parameter ω_{DE} versus the redshift z for model 2.

4. Conclusion

We investigated the cosmological application of the new gravitational scalar-tensor by considering two different models by choosing particular functional form of $\mathcal{K}$ and $\mathcal{G}$. We noticed from the plots for both models 1 and 2 that the evolution of the matter and dark energy density parameters Ω_m and Ω_{DE} shows the transition from the matter to the dark energy epoch. The evolution of the deceleration parameter q also show the transition from deceleration ($q > 0$) to acceleration ($q < 0$). The evolution of the dark-energy equation of state parameter ω_{DE} almost stabilizes in a value very close to the cosmological constant, however it seems to be affected by both the model we consider and the coupling constant ζ.

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